Homework 6

Instructions: Write your solutions on paper or a writing tablet, scan it and upload it to canvas. The file must be in pdf extension. Show neat and complete work and make sure that your scan is legible. Label your solutions and make sure they are in increasing order.

1 How to think of a matrix?

Say we have a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ denote the standard basis vectors.

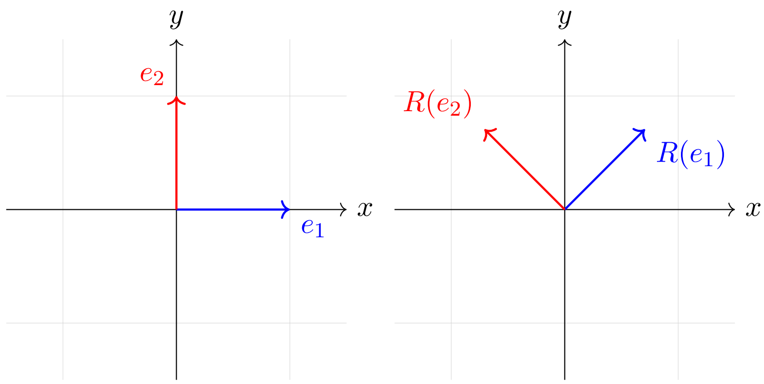
(a) Calculate Ae_1 and Ae_2.

(b) Define a linear transformation.

(c) Say $\vec{v} = 3e_1 + 4e_2$. What is $A\vec{v}$?

(d) Say $\vec{v} = c_1e_1 + c_2e_2$. What is $A\vec{v}$? (This should convince you that it suffices to define a linear transformation on the basis vectors. Where any vector $\vec{v}$ goes is just a linear combination of the images of e_1 and e_2).

(e) Let R denote the linear transformation for for rotation by $45°$. Write down the matrix for R.



2 Proving the theorem for general solution of a linear system with distinct eigenvalues

We will prove the following theorem from the book:

Theorem 1. *Take $\vec{x}' = P\vec{x}$ If P is an 2×2 constant matrix that has distinct real eigenvalues λ_1, λ_2, then there exist linearly independent corresponding eigenvectors $\vec{v_1}, \vec{v_2}$ and the general solution to can be written as*

$$\vec{x} = c_1 \vec{v_1} e^{\lambda_1 t} + c_2 \vec{v_2} e^{\lambda_2 t}$$

(a) First we do the easy case. Assume that P is a diagonal matrix. Then our system is

$$\vec{x}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \vec{x}$$

Find the general solution.(Note we cannot assume we know the theorem. We are trying to prove it).

(b) Now assume that $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Remark. *You cannot directly infer the eigenvalues from the matrix (they are not a and d).*

But, our hypothesis says that λ_1 and λ_2 are the distinct eigenvalues with corresponding eigenvectors v_1 and v_2. There is a theorem in Linear Algebra which says that any matrix with distinct eigenvalues

is diagonalizable:
Prove it:

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = B \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} B^{-1}$$

where $B = \begin{pmatrix} \vec{v_1} & \vec{v_2} \\ | & | \end{pmatrix}$.
(Hint: $Be_1 = \vec{v_1}$ and so, $B^{-1}v_1 = e_1$. Same equalities for e_2)

(c) Solve the system:

$$\vec{x}' = B \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} B^{-1}\vec{x}$$

(Note that you have proved the theorem)

3. Solve the following systems using eigenvalue method. First write down the matrix form.
   (a) Find the general solutions of $x_1' = 3x_1 + x_2$, $x_2' = 2x_1 + 4x_2$.
   (b) Find the general solutions of $x_1' = x_1 - 2x_2$, $x_2' = 2x_1 + x_2$.
   (c) Find the general solutions when $P = \begin{pmatrix} 9 & -2 & -6 \\ -8 & 3 & 6 \\ 10 & -2 & -6 \end{pmatrix}$

4. For each of the following systems
   - find the eigenvalues
   - draw the vector field and label the eigenvectors
   - draw a few solution curves

   (No need to solve the system).

   (a) $x' = x + y$, $y' = x - y$
   (b) $x_1' = x_1 + x_2$, $x_2' = 3x_2$
   (c) $x_1' = -3x_2$, $x_2' = 3x_1$
   (d) $x' = x + 3y$, $y' = -2x - 4y$

(e) $x' = x + 3y$, $y' = -2x - 4y$

(f) $x' = -2x - 6y$, $y' = -8x + 6y$

(g) $x' = -3x + y$, $y' = x - 4y$

Remark. *Try to draw it from first principles. Do not just copy the diagram from the book.*